ON MEASURES ON PARTITIONS ARISING IN HARMONIC ANALYSIS FOR LINEAR AND PROJECTIVE CHARACTERS OF THE INFINITE SYMMETRIC GROUP

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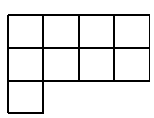
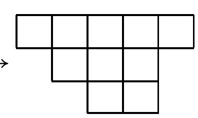
Abstract. The z-measures on partitions originated from the problem of harmonic analysis of linear representations of the infinite symmetric group in the works of Kerov, Olshanski and Vershik [KOV93], [KOV04]. A similar family corresponding to projective representations was introduced by Borodin [Bor99]. The latter measures live on strict partitions (i.e., partitions with distinct parts), and the z-measures are supported by all partitions. In this note we describe some combinatorial relations between these two families of measures using the well-known doubling of shifted Young diagrams.

1. Ordinary and strict partitions

A *partition* is an integer sequence of the form $\rho = (\rho_1 \ge \ldots \ge \rho_{\ell(\rho)}, 0, 0, \ldots)$, where each $\rho_i > 0$ and only finitely many of them are nonzero. A partition is called *strict* if all its nonzero parts are distinct. Strict partitions are denoted by $\lambda, \mu, \ldots$. Partitions which are not necessary strict will be called *ordinary* and denoted by $\rho, \sigma, \ldots$. We denote $|\rho| := \rho_1 + \cdots + \rho_{\ell(\rho)}$, this is the *weight* of a partition. Set $\mathbb{Y}_n := \{\rho : |\rho| = n\}$, $\mathbb{S}_n := \{\lambda : \lambda \text{ strict and } |\lambda| = n\}$, $n = 0, 1, 2, \ldots$ (by agreement, $\mathbb{Y}_0 = \mathbb{S}_0 = \{\varnothing\}$).

We identify ordinary and strict partitions with corresponding *ordinary* and *shifted Young diagrams*, respectively [Mac95, I.1]. For example:

$\rho = (4,4,1) \longleftrightarrow$  $\lambda = (5,3,2) \longleftrightarrow$  (1)

For any box $\square$ in an ordinary of shifted Young diagram, by $\mathsf{i}(\square)$ and $\mathsf{j}(\square)$ we denote its row and column numbers, respectively. Also $\mathsf{c}(\square) := \mathsf{j}(\square) - \mathsf{i}(\square)$ is the *content* of the box. Clearly, the content of any box in a shifted Young diagram is nonnegative.

If we have $|\rho| = |\sigma| + 1$ and $\sigma \subset \rho$ for ordinary Young diagrams σ, ρ (i.e., ρ is obtained from σ by adding a box), then we write $\sigma \nearrow \rho$, or, equivalently, $\rho \searrow \sigma$. In a similar situation for shifted diagrams μ, λ we write $\mu \nearrow\!\!\!\nearrow \lambda$ or $\lambda \searrow\!\!\!\searrow \mu$.

The *Young graph* $\mathbb{Y} = \bigsqcup_{n=0}^{\infty} \mathbb{Y}_n$ consists of all ordinary Young diagrams, and we connect $\sigma \in \mathbb{Y}_{n-1}$ and $\rho \in \mathbb{Y}_n$ by an edge iff $\sigma \nearrow \rho$. This is a graded graph which describes the branching of irreducible representations of the symmetric groups $\mathfrak{S}(n)$, see [Mac95, I.7] or [OV96]. The *Schur graph* $\mathbb{S} = \bigsqcup_{n=0}^{\infty} \mathbb{S}_n$ is defined in the same

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manner for shifted Young diagrams. This graded graph describes the branching of (suitably normalized) irreducible truly projective characters of the symmetric groups $\mathfrak{S}(n)$ [HH92], [Iva99].

For $\rho \in \mathbb{Y}$, by f_ρ denote the number of paths in $\mathbb{Y}$ from the initial vertex $\varnothing$ to the diagram ρ. The number of paths in the Schur graph from $\varnothing$ to $\lambda \in \mathbb{S}$ is denoted by g_λ. There are explicit formulas for f_ρ and g_λ [Mac95, I.5, III.8].

2. Coherent systems of measures

2.1. **Young graph.** *Down transition probabilities* on the Young graph are

$$p^{\downarrow}(\rho,\sigma) := \begin{cases} f_\sigma/f_\rho, & \text{if } \sigma \nearrow \rho, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

One sees that $p^{\downarrow}$ give rise to a Markov transition kernel from $\mathbb{Y}_n$ to $\mathbb{Y}_{n-1}$ (for any $n = 1, 2, \dots$), i.e., to a random procedure of deleting a box from an ordinary Young diagram.

A sequence of probability measures $\{\mathrm{M}_n\}$ on $\mathbb{Y}_n$ is called *coherent* iff M_n is compatible with the down transition kernel $p^{\downarrow}$: $\mathrm{M}_n \circ p^{\downarrow} = \mathrm{M}_{n-1}$, or, in more detail,

$$\sum_{\rho:\rho\searrow\sigma} \mathrm{M}_n(\rho) p^{\downarrow}(\rho,\sigma) = \mathrm{M}_{n-1}(\sigma) \quad \text{for all } n = 1, 2, \dots, \text{ and } \sigma \in \mathbb{Y}_{n-1}.$$

We assume that our coherent systems are *nondegenerate*, i.e., each M_n is supported by the whole $\mathbb{Y}_n$. Having a nondegenerate coherent system $\{\mathrm{M}_n\}$, one can define the corresponding up transition kernel from $\mathbb{Y}_n$ to $\mathbb{Y}_{n+1}$ (for any $n = 0, 1, \dots$):

$$p^{\uparrow}(\sigma,\rho) := \begin{cases} p^{\downarrow}(\rho,\sigma)\mathrm{M}_{n+1}(\rho)/\mathrm{M}_n(\sigma), & \text{if } \sigma \nearrow \rho \text{ and } |\sigma| = n, \\ 0, & \text{otherwise.} \end{cases}$$

The up transition probabilities depend on the choice of a coherent system $\{\mathrm{M}_n\}$, and they define it uniquely. Moreover, $\mathrm{M}_n \circ p^{\uparrow} = \mathrm{M}_{n+1}$ for all n. In this way, $p^{\uparrow}$ define a random procedure of adding a box to a Young diagram. Iterating this procedure, one can think of a process of random growth of a diagram (by adding one box at a time) which starts from $\varnothing$. Then M_n is the distribution of a Young diagram after adding n boxes. It is known (e.g., see [VK87]) that linear characters of the infinite symmetric group $\mathfrak{S}(\infty)$ are in one-to-one correspondence with coherent systems on the Young graph.

The well-known Plancherel measures on ordinary partitions $\mathrm{Pl}_n(\rho) = f_\rho^2/n!$ ($\rho \in \mathbb{Y}_n$) form a distinguished coherent system $\{\mathrm{Pl}_n\}$ on the Young graph. It has the up transition probabilities $p^{\uparrow}_{\mathrm{Pl}}(\sigma,\rho) = \frac{f_\rho}{(|\sigma|+1)f_\sigma}$ ($\sigma \nearrow \rho$).

The problem of harmonic analysis on the infinite symmetric group [KOV93], [KOV04] leads to a deformation $\{\mathrm{M}_n^{z,z'}\}$ of the Plancherel measures Pl_n depending on two complex parameters z and z' subject to the following constraints:

- either $z' = \bar{z}$ and $z \in \mathbb{C} \setminus \mathbb{Z}$,
- or $z, z' \in \mathbb{R}$ and $m < z, z' < m+1$ for some $m \in \mathbb{Z}$.

The system of deformed measures $\{\mathrm{M}_n^{z,z'}\}$ (they are called the *z-measures*) is also coherent, and its up transition probabilities have the form (e.g., see [Ker00]):

$$p^{\uparrow}_{z,z'}(\sigma,\rho) = \frac{(z + \mathsf{c}(\square))(z' + \mathsf{c}(\square))}{zz' + |\sigma|} p^{\uparrow}_{\mathrm{Pl}}(\sigma,\rho), \qquad \sigma \nearrow \rho, \quad \square = \rho \setminus \sigma. \tag{3}$$

The z-measures $\{M_n^{z,z'}\}$ is a remarkable object, they were studied in great detail by Borodin, Olshanski, Okounkov, and other authors.

2.2. **Schur graph.** General concepts explained above in the case of the Young graph work in the same way for the Schur graph. The down transition probabilities here are denoted by $p^{\Downarrow}$, they are defined as in (2) using the quantities g_λ. Coherent systems of measures on the Schur graph correspond to (truly) projective characters of $\mathfrak{S}(\infty)$ (e.g., see [Naz92], [Iva99] and a general formalism of [VK87]).

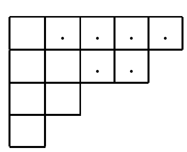
There are also Plancherel measures on strict partitions $\mathbb{P}\mathbb{l}_n(\lambda) = 2^{n-\ell(\lambda)}g_\lambda^2/n!$ (here $\lambda \in \mathbb{S}_n$ and $\ell(\lambda)$ is the number of rows in the shifted diagram λ), they form a distinguished coherent system on $\mathbb{S}$. The corresponding up transition probabilities are $p^{\Uparrow}_{\mathbb{P}\mathbb{l}}(\mu,\lambda) = \frac{g_\lambda}{(|\mu|+1)g_\mu}2^{\ell(\mu)-\ell(\lambda)+1}$ ($\mu \nearrow\!\!\!\nearrow \lambda$). A deformation $\{\mathbb{M}_n^\alpha\}$ of the Plancherel measures $\mathbb{P}\mathbb{l}_n$ depending on one parameter $\alpha > 0$ was introduced in [Bor99]. The measures $\mathbb{M}_n^\alpha$ form a coherent system which can be described in terms of its up transition probabilities:

$$p^{\Uparrow}_{\alpha}(\mu,\lambda) = \frac{\mathsf{c}(\square)\cdot(\mathsf{c}(\square)+1)+\alpha}{2|\mu|+\alpha}p^{\Uparrow}_{\mathbb{P}\mathbb{l}}(\mu,\lambda), \qquad \mu \nearrow\!\!\!\nearrow \lambda, \quad \square = \lambda\setminus\mu. \tag{4}$$

The Plancherel measures on Young and Schur graphs admit a unified combinatorial description which can be read from, e.g., [Fom94]. The measures $\{\mathbb{M}_n^\alpha\}$ on the Schur graph do not have a representation-theoretic interpretation in the spirit of [KOV93], [KOV04] yet. However, combinatorially they look very similar to the z-measures: the families $M_n^{z,z'}$ and $\mathbb{M}_n^\alpha$ can be characterized in a unified manner, see [Roz99], [Bor99]; see also [Pet10c, §4.1] for another characterization. On the other hand, most results about $\mathbb{M}_n^\alpha$ do not follow directly from the corresponding results about the z-measures. In this paper we aim to describe certain *direct* combinatorial relations between $M_n^{z,z'}$ and $\mathbb{M}_n^\alpha$, and, more general, between the Young and the Schur graphs. There are also other aspects in which $M_n^{z,z'}$ and $\mathbb{M}_n^\alpha$ are directly related, e.g., at the level of correlation kernels of corresponding random point processes, see [Pet10c, (7.17), (8.3)], and [Pet10b, Remark 6].

3. Doubling of shifted Young diagrams and down transition probabilities

Let $\lambda = (\lambda_1,\dots,\lambda_\ell)$ be a shifted Young diagram. By $\mathcal{D}\lambda$ let us denote its *doubling*, i.e., the ordinary Young diagram with $2|\lambda|$ boxes which has Frobenius coordinates $(\lambda_1,\dots,\lambda_\ell \mid \lambda_1-1,\dots,\lambda_\ell-1)$ [Mac95, I.1]. E.g., for $\lambda = (4,2)$ we have

$\mathcal{D}\lambda =$ 

(the original shifted diagram is marked). In this way, $\mathcal{D}$ defines an embedding of $\mathbb{S}$ into $\mathbb{Y}$. Any ordinary Young diagram of the form $\mathcal{D}\lambda$ will be called $\mathcal{D}$*-symmetric*. A finite path in the Young graph $\varnothing \nearrow \rho^{(1)} \nearrow \cdots \nearrow \rho^{(k)}$ is called a $\mathcal{D}$*-path* iff each Young diagram $\rho^{(2m)}$ is $\mathcal{D}$-symmetric. The next statement is straightforward:

Lemma 1. *Let $\mu \nearrow\!\!\!\nearrow \lambda$ be two shifted Young diagrams. If $\ell(\lambda) = \ell(\mu)$, there exist two ordinary diagrams $\rho^{(a),(b)}$ such that $\mathcal{D}\mu \nearrow \rho^{(a),(b)} \nearrow \mathcal{D}\lambda$. If $\ell(\lambda) = \ell(\mu)+1$,*

there is only one such ordinary diagram ρ. Consequently, for any $\lambda \in \mathbb{S}$ the number of $\mathcal{D}$-paths in $\mathbb{Y}$ from $\varnothing$ to $\mathcal{D}\lambda$ is $2^{|\lambda|-\ell(\lambda)} g_\lambda$.

Note that the ambient structure of $\mathcal{D}$-paths in $\mathbb{Y}$ defines certain edge multiplicities in $\mathcal{D}\mathbb{S} \subset \mathbb{Y}$ (and, therefore, in $\mathbb{S}$): there are either one (if $\ell(\lambda) = \ell(\mu) + 1$) or two (if $\ell(\lambda) = \ell(\mu)$) edges between shifted diagrams $\mu \nearrow\!\!\!\nearrow \lambda$. However, these new edge multiplicities give rise to the same down transition probabilities $p^{\Downarrow}$ on $\mathbb{S}$ as before. Following [Ker89], one can say that the new edge multiplicities are *equivalent* to the old ones.

Proposition 2. *In notation of Lemma 1, if $\ell(\lambda) = \ell(\mu)$, one has $p^{\Downarrow}(\lambda,\mu) = p^{\downarrow}(\mathcal{D}\lambda, \rho^{(a)}) + p^{\downarrow}(\mathcal{D}\lambda, \rho^{(b)})$, and if $\ell(\lambda) = \ell(\mu) + 1$, then $p^{\Downarrow}(\lambda,\mu) = p^{\downarrow}(\mathcal{D}\lambda, \rho)$.*

Proof. Fix $\lambda \in \mathbb{S}_n$ and $\mu \in \mathbb{S}_{n-1}$ such that $\mu \nearrow\!\!\!\nearrow \lambda$. Assume that $\ell(\lambda) = \ell(\mu)$, the other case is similar. Down transition probabilities $p^{\downarrow}$ on the Young graph allow to define a Markov chain going down from $\mathcal{D}\lambda$, i.e., a sequence of random ordinary Young diagrams $\mathcal{D}\lambda = \boldsymbol{\varrho}_0 \searrow \boldsymbol{\varrho}_1 \searrow \dots \searrow \boldsymbol{\varrho}_{2n} = \varnothing$. For each k, the conditional distribution of $\boldsymbol{\varrho}_k$ given $\boldsymbol{\varrho}_{k-1}$ is governed by the transition kernel $p^{\downarrow}$ from $\mathbb{Y}_{2n-k+1}$ to $\mathbb{Y}_{2n-k}$. In other words, this gives a measure on the set of all paths in $\mathbb{Y}$ from $\varnothing$ to $\mathcal{D}\lambda$. By the very definition of $p^{\downarrow}$ (2), this measure is uniform over all such paths. Let D denote the event that the path $(\boldsymbol{\varrho}_{2n} \nearrow \dots \nearrow \boldsymbol{\varrho}_0)$ from $\varnothing$ to $\mathcal{D}\lambda$ is a $\mathcal{D}$-path. Conditioning on the event D, we have a uniform measure over $\mathcal{D}$-paths. One clearly has

$$\mathsf{Prob}(\boldsymbol{\varrho}_2 = \mathcal{D}\mu, D) = \mathsf{Prob}(\boldsymbol{\varrho}_1 = \rho^{(a)}, D) + \mathsf{Prob}(\boldsymbol{\varrho}_1 = \rho^{(b)}, D). \tag{5}$$

In the left-hand side one has $(f_{\mathcal{D}\lambda})^{-1}$ times the number of $\mathcal{D}$-paths in $\mathbb{Y}$ from $\varnothing$ to $\mathcal{D}\lambda$ which also go through $\mathcal{D}\mu$, and in the right-hand side the events $\{\boldsymbol{\varrho}_1 = \rho^{(a),(b)}\}$ are independent of D, and $\mathsf{Prob}(\boldsymbol{\varrho}_1 = \rho^{(a),(b)}) = p^{\downarrow}(\mathcal{D}\lambda, \rho^{(a),(b)})$. Dividing (5) by $\mathsf{Prob}(D) = 2^{|\lambda|-\ell(\lambda)} g_\lambda / f_{\mathcal{D}\lambda}$, we get the desired identity. □

4. Plancherel up transition probabilities

Here we describe an identity for the Plancherel up transition probabilities $p^{\uparrow}_{\mathrm{Pl}}$ and $p^{\Uparrow}_{\mathbb{P}\mathbb{I}}$ which is "dual" to Proposition 2. The proof uses Kerov's interlacing coordinates of ordinary and shifted Young diagrams [Ker00], [Ols10], [Pet10a]. Let us recall necessary definitions and facts from these papers.

For an ordinary Young diagram ρ, by $\mathrm{x}_1, \dots, \mathrm{x}_d$ and $\mathrm{y}_1, \dots, \mathrm{y}_{d-1}$ denote the contents of all boxes that can be added to or removed from ρ, respectively. It is known that these numbers interlace ($\mathrm{x}_1 < \mathrm{y}_1 < \mathrm{x}_2 < \dots < \mathrm{y}_{d-1} < \mathrm{x}_d$) and define ρ uniquely. For example, for $\rho = (4,4,1)$ (see (1)) one has $d = 3$, $(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3) = (-3,-1,4)$, and $(\mathrm{y}_1, \mathrm{y}_2) = (-2, 2)$. The Plancherel up transition probabilities for $\mathbb{Y}$ arise as the following coefficients in the expansion as a sum of partial fractions:

$$\mathcal{R}^{\uparrow}(u;\rho) := \frac{(u-\mathrm{y}_1)\dots(u-\mathrm{y}_{d-1})}{(u-\mathrm{x}_1)\dots(u-\mathrm{x}_{d-1})(u-\mathrm{x}_d)} = \sum_{s=1}^{d} \frac{p^{\uparrow}_{\mathrm{Pl}}(\rho; \rho + \boxed{\mathrm{x}_s})}{u - \mathrm{x}_s}.$$

Here $\rho + \boxed{\mathrm{x}_s}$ means that we add to ρ a box with content x_s.

The case of shifted diagrams is slightly more complicated, and in full detail it is explained in [Pet10a, §3] (the arXiv version). Let λ be a shifted Young diagram. Let $\mathbb{y}_1, \dots, \mathbb{y}_k$ denote the contents of all boxes that can be removed from λ. Let $\mathbb{x}_1, \dots, \mathbb{x}_k$ denote all the *nonzero* contents of all boxes that can be added to λ. These contents also interlace ($\mathbb{y}_1 < \mathbb{x}_1 < \mathbb{y}_2 < \dots < \mathbb{y}_k < \mathbb{x}_k$) and define λ uniquely. For example,

for $\lambda = (5,3,2)$ (see (1)) one has $k = 2$, $(\mathbbm{x}_1, \mathbbm{x}_2) = (3,5)$, and $(\mathbbm{y}_1, \mathbbm{y}_2) = (1,4)$. The Plancherel up transition probabilities for $\mathbb{S}$ arise as the following expansion coefficients:

$$\mathcal{R}^{\Uparrow}(v;\lambda) := \frac{(v - \mathbbm{y}_1(\mathbbm{y}_1+1))\dots(v-\mathbbm{y}_k(\mathbbm{y}_k+1))}{v(v-\mathbbm{x}_1(\mathbbm{x}_1+1))\dots(v-\mathbbm{x}_k(\mathbbm{x}_k+1))} = \sum_{\mathbbm{x}} \frac{p^{\Uparrow}_{\mathbb{P}\mathbb{l}}(\lambda;\lambda+\boxed{\mathbbm{x}})}{v-\mathbbm{x}(\mathbbm{x}+1)},$$

where the sum is taken over *all* boxes which can be added to λ, and $\mathbbm{x}$ is the content of such a box (here it does not have to be nonzero).

The next fact is readily checked:

Proposition 3. *For any $\lambda \in \mathbb{S}$, one has $(u-1)\cdot\mathcal{R}^{\Uparrow}(u(u-1);\lambda) = \mathcal{R}^{\uparrow}(u;\mathcal{D}\lambda)$. Consequently, in notation of Lemma 1, $p^{\Uparrow}_{\mathbb{P}\mathbb{l}}(\mu,\lambda) = p^{\uparrow}_{\mathrm{Pl}}(\mathcal{D}\mu,\rho^{(a)}) + p^{\uparrow}_{\mathrm{Pl}}(\mathcal{D}\mu,\rho^{(b)})$ for $\ell(\lambda)=\ell(\mu)$, and $p^{\Uparrow}_{\mathbb{P}\mathbb{l}}(\mu,\lambda) = p^{\uparrow}_{\mathrm{Pl}}(\mathcal{D}\mu,\rho)$ otherwise.*

Proposition 2 can also be proved using the above rational functions because the down transition probabilities essentially arise as coefficients of expansions of $1/\mathcal{R}^{\uparrow}(u;\rho)$ and $1/(v\cdot\mathcal{R}^{\Uparrow}(v;\lambda))$.

5. Up transition probabilities for $\mathrm{M}_n^{z,z'}$ and $\mathbb{M}_n^{\alpha}$

By suitable choice of the parameters z, z' of the z-measures on ordinary partitions, one can get an analogue of Proposition 3 for the deformed coherent systems $\mathrm{M}_n^{z,z'}$ and $\mathbb{M}_n^{\alpha}$, which is the main result of the present note. Set $\nu(\alpha) := \frac{1}{2}\sqrt{1-4\alpha}$. From (3), (4) and Proposition 3 we have:

Proposition 4. *Let $z(\alpha) = \nu(\alpha) - \frac{1}{2}$, $z'(\alpha) = -\nu(\alpha) - \frac{1}{2}$ (note that these parameters are admissible for the z-measures). In notation of Lemma 1, for $\ell(\lambda) = \ell(\mu)$ one has $p^{\Uparrow}_{\alpha}(\mu,\lambda) = p^{\uparrow}_{z(\alpha),z'(\alpha)}(\mathcal{D}\mu,\rho^{(a)}) + p^{\uparrow}_{z(\alpha),z'(\alpha)}(\mathcal{D}\mu,\rho^{(b)})$, and if $\ell(\lambda) = \ell(\mu)+1$, then one has $p^{\Uparrow}_{\alpha}(\mu,\lambda) = p^{\uparrow}_{z(\alpha),z'(\alpha)}(\mathcal{D}\mu,\rho)$.*

Now one can explain how the random growth processes for the measures $\mathbb{M}_n^{\alpha}$ and $\mathrm{M}_n^{z(\alpha),z'(\alpha)}$ are related. Indeed, to grow a random *shifted* Young diagram $\boldsymbol{\lambda}$ with n boxes distributed according to $\mathbb{M}_n^{\alpha}$, one should start the growth process on the *Young graph* from $\varnothing$ which evolves as follows:

- at each *even* step add a box to the ordinary diagram according to the probabilities $p^{\uparrow}_{z(\alpha),z'(\alpha)}$ (this is a random procedure);
- at each *odd* step add the unique box to the current ordinary diagram so that it again becomes $\mathcal{D}$-symmetric (this is a deterministic procedure).

In this way the growth process on the Young graph goes along a $\mathcal{D}$-symmetric path, and after $2n$ steps it reaches a random ordinary Young diagram $\mathcal{D}\boldsymbol{\lambda} \in \mathbb{Y}_{2n}$, where $\boldsymbol{\lambda} \in \mathbb{S}_n$ is distributed according to $\mathbb{M}_n^{\alpha}$. One may call this the *forced $\mathcal{D}$-symmetrization* of the old growth process (3) on the Young graph: the growing ordinary Young diagram is forced to be $\mathcal{D}$-symmetric at every step at which it is possible.

6. Schur measures and an analogue for shifted diagrams

Both families of measures that we consider can be interpreted through certain specializations of Schur symmetric functions s_τ, $\tau \in \mathbb{Y}$ [Mac95, I.3]. For the z-measures one has [Oko01]

$$\mathrm{M}_n^{z,z'}(\rho) = \frac{n!}{(zz')_n} s_\rho(\underbrace{1,\dots,1}_{z \text{ times}}) s_\rho(\underbrace{1,\dots,1}_{z' \text{ times}}), \qquad \rho \in \mathbb{Y}_n$$

(here $(\cdots)_n$ denotes the Pochhammer symbol). For the measures $\mathbb{M}_n^\alpha$ one can show that (see also [Pet10b, §2.6])

$$\mathbb{M}_n^\alpha(\lambda) = \frac{(-1)^n n!}{(\alpha/2)_n} s_{\mathcal{D}\lambda}(\underbrace{1,1,\dots,1,1}_{\nu(\alpha)-\frac{1}{2} \text{ times}}), \qquad \lambda \in \mathbb{S}_n.$$

Such measures were first considered in [Rai00, Thm 7.1].

From the above two formulas one sees that the weights $\{\mathbb{M}_n^\alpha(\lambda)\}_{\lambda\in\mathbb{S}_n}$ are proportional (with a coefficient depending only on n) to square roots of the weights $\{\mathrm{M}_{2n}^{z(\alpha),z'(\alpha)}(\mathcal{D}\lambda)\}_{\lambda\in\mathbb{S}_n}$. (Alternatively, this can be seen from §5 and the multiplicative nature of our measures [Roz99], [Bor99].) This property can easily be reformulated in probabilistic terms, but it does not seem to provide a direct way of obtaining properties of $\mathbb{M}_n^\alpha$ from the corresponding properties of the z-measures.

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